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Quasi-interpolants Based on Trigonometric Splines

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DEDICATED TO THE MEMORY OF PER ERIK KOCH†

A general theory of quasi-interpolants based on trigonometric splines is developed which is analogous to the polynomial spline case. The aim is to construct quasi-interpolants which are local, easy to compute, and which apply to a wide class of functions. As examples, we give a detailed treatment including error bounds for two classes which are especially useful in practice.

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1. INTRODUCTION

Among the many different generalizations of polynomial splines, the trigonometric splines are of particular theoretical interest and practical importance. They were introduced in [12], and have been studied in a long list of papers which we do not cite here, see [3, 6, 9, 10] and references therein. The purpose of this paper is to develop a general theory of trigonometric quasi-interpolants of the form $Qf = \sum (\lambda_i f)\, T_i^k$, where the T_i^k are certain trigonometric B-splines, and the λ_i are appropriate linear functionals chosen so that:

(1) Q can be applied to a wide class of functions including, for example, continuous functions,

(2) the coefficients $\lambda_i f$ of the quasi-interpolant can be computed directly from information on f without solving systems of equations,

(3) Qf is *local* in the sense that $Qf(x)$ depends only on the values of f in a small neighborhood of x,

(4) if f is a smooth function, Qf provides an optimal order approximation to f (i.e., of the same order as the best trigonometric spline approximation).

In addition to developing a general theory, we give a detailed treatment of two interesting classes of quasi-interpolants based on derivative information and on simple point evaluation. In both cases we establish error bounds, and pay special attention to the associated constants, and in particular how they depend on certain mesh ratios.

While the analysis here parallels the treatment in [8] of quasi-interpolants based on polynomial splines, because of the nature of trigonometric splines, the details are considerably more complicated.

The paper is organized as follows. We begin by recalling some basic facts about trigonometric polynomials and trigonometric splines in Section 2. In Section 3 we develop a general theory of quasi-interpolants based on trigonometric splines. In Section 4 we discuss several trigonometric Taylor expansions, and use them as a tool to derive a general error bound. In Section 5 we recall some results on trigonometric blossoming, and apply them to establish some general Marsden identities for trigonometric splines. Quasi-interpolants based on derivatives and on point evaluators are treated in Sections 6 and 7, respectively. Detailed error bounds for these quasi-interpolants, including both local and global results can be found in Sections 8–9. The question of how the constants in the error bounds for the derivative operator depend on mesh ratios is dealt with in Section 10. Finally, the last section of the paper is devoted to several remarks.

2. TRIGONOMETRIC SPLINES

Let $s(x) := \sin(x/2)$, $c(x) := \cos(x/2)$. Given a positive integer k, let

$$\mathcal{T}_k := \begin{cases} \operatorname{span}\{1, s(2x), c(2x), s(4x), c(4x), ..., s((k-1)\,x), c((k-1)\,x)\}, \\ \qquad k \text{ odd} \\ \operatorname{span}\{s(x), c(x), s(3x), c(3x), ..., s((k-1)\,x), c((k-1)\,x)\}, \\ \qquad k \text{ even}, \end{cases}$$

be the *space of trigonometric polynomials of order* k. We observe that $\mathcal{T}_l \subset \mathcal{T}_k$ if $k-l \geqslant 0$ is even, but not if it is odd. Suppose

$$\Delta := \{a = x_0 < x_1 < \cdots < x_m < x_{m+1} = b\}$$

is a partition of the interval $J := [a, b]$ into $m+1$ subintervals. Let $\mathcal{K} = (k_1, ..., k_m)$ be a vector of integers satisfying $1 \leqslant k_i \leqslant k$, $i = 1, ..., m$. Then the associated *space of trigonometric splines of order* k is defined [10] by

$$\begin{aligned} \mathcal{S} &:= \mathcal{S}(\mathcal{T}_k; \mathcal{K}; \Delta) \\ &:= \{g : g|_{(x_i, x_{i+1})} \in \mathcal{T}_k,\ i = 0, ..., m, \text{ and } D_-^{j-1} g(x_i) = D_+^{j-1} g(x_i), \\ &\qquad j = 1, ..., k - k_i,\ i = 1, ..., m\}. \end{aligned}$$

It is well known that $\dim \mathcal{S}(\mathcal{T}_k; \mathcal{K}; \Delta) = n := k + \sum_{i=1}^m k_i$. Following [10], to construct a basis of locally supported splines spanning $\mathcal{S}(\mathcal{T}_k; \mathcal{K}; \Delta)$, we introduce the *extended knot sequence*

$$t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n+k}, \tag{2.1}$$

where

$$a = t_1 = \cdots = t_k, \qquad t_{n+1} = \cdots = t_{n+k} = b,$$

and $\{t_{k+1} \leqslant \cdots \leqslant t_n\}$ is the set obtained by repeating each x_i a total of k_i times, $i = 1, ..., m$. Throughout this paper we will assume that the knots are such that

$$0 < t_{i+k-1} - t_i < 2\pi, \qquad i = 1, ..., n. \tag{2.2}$$

Associated with the extended partition, let

$$T_i^1(x) := \begin{cases} 1, & \text{if } \ t_i \leqslant x < t_{i+1} \\ 0, & \text{otherwise}, \end{cases}$$

and for $k > 1$, let

$$T_i^k(x) := \frac{s(x - t_i)}{s(t_{i+k-1} - t_i)} T_i^{k-1}(x) + \frac{s(t_{i+k} - x)}{s(t_{i+k} - t_{i+1})} T_{i+1}^{k-1}(x). \tag{2.3}$$

Here T_i^k is defined to be identically zero if $t_{i+k} = t_i$, and terms in (2.3) with zero denominator are treated as zero.

The T_i^k are the well-known *trigonometric B-splines*, see [9, 10]. The set $\{T_i^k\}_{i=1}^n$ is a basis for $\mathcal{S}$. Moreover, each $T_i^k(x)$ is positive for $x \in (t_i, t_{i+k})$, and is zero for all $x \notin [t_i, t_{i+k}]$.

3. TRIGONOMETRIC QUASI-INTERPOLANTS

Given an integer $k \geqslant 1$, let $\{T_i^k\}_{i=1}^n$ be the set of trigonometric B-splines spanning the space $\mathcal{S}$ as in the previous section.

DEFINITION 3.1. Let $\lambda_1, ..., \lambda_n$ be a set of linear functionals which are defined on a space of functions $\mathcal{F}$ defined on the interval $J = [a, b]$ with $\mathcal{S} \subset \mathcal{F}$. Then for any $f \in \mathcal{F}$,

$$Qf := \sum_{i=1}^{n} (\lambda_i f)\, T_i^k \tag{3.1}$$

is called a *trigonometric quasi-interpolant of f.*

Clearly, the properties of the quasi-interpolant Q are determined by the choice of the linear functionals $\{\lambda_i\}_{i=1}^n$. We are interested in the following questions:

(1) *What is the class of functions $\mathcal{F}$ to which Q can be applied*? To get a quasi-interpolant which applies to continuous functions on J, we can define $\lambda_i f$ to be a linear combination of values of f at points in J. Alternatively, we can build Qf from derivatives of f (which restricts the applicability of Q), or from integrals of f (which extends its applicability).

(2) *When is Q local*? By the support properties of the trigonometric B-splines, given $t_m \leqslant x < t_{m+1}$, the only B-splines which are nonzero at x are $T_{m+1-k}^k, ..., T_m^k$. Thus, for example, we can get a local method by making $\lambda_i f$ depend only on the values of f on the support interval $[t_i, t_{i+k}]$ of T_i^k for each $i = 1, ..., n$.

(3) *How well does Qf approximate smooth functions f*? In order to make Qf approximate smooth functions f well, we shall construct Q such that

$$Qf = f, \qquad \text{all} \quad f \in \mathcal{T}_l \tag{3.2}$$

for some $1 \leqslant l \leqslant k$. The higher we can make l, the better approximation properties Q will have.

We devote the remainder of this section to the question of how to construct quasi-interpolants of the form

$$Qf := \sum_{i=1}^{n} \sum_{j=1}^{l} \alpha_{i,j} \lambda_{i,j} f \, T_i^k \tag{3.3}$$

satisfying (3.2), where $\{\lambda_{i,j}\}_{i,j=1}^{n,l}$ are prescribed linear functionals.

LEMMA 3.2. *Fix $1 \leqslant l \leqslant k$ with $k-l$ even. Let $\{p_\nu\}_{\nu=1}^{l}$ be any basis for the space $\mathscr{T}_l$, and suppose that for each $1 \leqslant i \leqslant n$, $\{\lambda_{i,1}, ..., \lambda_{i,l}\}$ is a set of linear functionals such that*

$$\det(\lambda_{i,j} p_\nu)_{j,\nu=1}^{l} \neq 0. \tag{3.4}$$

Then there is a unique set of coefficients $\{\alpha_{i,j}\}$ so that the operator Q defined in (3.3) *satisfies* (3.2).

Proof. Clearly, Q reproduces $\mathscr{T}_l$ if and only if it reproduces $p_1, ..., p_l$. If

$$p_\nu(x) = \sum_{i=1}^{n} b_{\nu,i} T_i^k, \tag{3.5}$$

then $Qp_\nu = p_\nu$ is equivalent to

$$Qp_\nu - p_\nu = \sum_{i=1}^{n} (\lambda_i p_\nu - b_{\nu,i}) \, T_i^k = 0.$$

By the linear independence of the T_i^k, we conclude that Q reproduces $\mathscr{T}_l$ if and only if for each $i = 1, ..., n$, the coefficients $\{\alpha_{i,j}\}_{j=1}^{l}$ solve the system

$$\lambda_i p_\nu = \sum_{j=1}^{l} \alpha_{i,j} \lambda_{i,j} p_\nu = b_{\nu,i}, \qquad \nu = 1, ..., l. \tag{3.6}$$

By (3.4), each of these systems has a unique solution, and the proof is complete. ∎

There is no analog of this lemma for $k-l$ odd since the trigonometric polynomials $\mathscr{T}_l$ are not contained in the spline space $\mathscr{S}$ for such l. To use the lemma in practice, we need to find some trigonometric polynomials $p_1, ..., p_l$ which satisfy (3.4) and whose B-spline expansions are known. Then for each $1 \leqslant i \leqslant n$, we can set up the system (3.6) and solve it numerically for the coefficients $\alpha_{i,1}, ..., \alpha_{i,l}$. We can save the work of solving these systems by choosing the p_ν so that the matrix in (3.6) reduces to the identity matrix.

LEMMA 3.3. *Suppose the hypotheses of Lemma 3.2 hold. For each* $1 \leqslant i \leqslant n$, *let* $p_{i,1}, ..., p_{i,l}$ *be the unique trigonometric polynomials in* $\mathscr{T}_l$ *such that*

$$\lambda_{i,\nu} p_{i,j} = \delta_{\nu,j}, \qquad j, \nu = 1, ..., l. \tag{3.7}$$

Then the quasi-interpolant Q defined in (3.3) *satisfies* (3.2) *if and only if*

$$\alpha_{i,j} = b_{i,j,i}, \qquad j = 1, ..., l, \tag{3.8}$$

where

$$p_{i,j} = \sum_{\mu=1}^{n} b_{i,j,\mu} T_\mu^k. \tag{3.9}$$

Proof. Lemma 3.2 guarantees the existence of unique $p_{i,j}$ satisfying (3.7), while the fact that $p_{i,j} \in \mathscr{T}_l \subset \mathscr{S}$ assures the existence of unique $b_{i,j,\mu}$ such that (3.9) holds. Then by the proof of Lemma 3.2, $Qp_{i,\nu} = p_{i,\nu}$ implies

$$\sum_{j=1}^{l} \alpha_{i,j} \lambda_{i,j} p_{i,\nu} = b_{i,\nu,i}, \qquad \nu = 1, ..., l.$$

In view of (3.7), this implies that the unique coefficients which make Q satisfy (3.2) are given by (3.8). ∎

We can use blossoming (see Section 5 below) to find explicit formulae for the coefficients $\alpha_{i,j}$. This leads to

THEOREM 3.4. *Let* $1 \leqslant l \leqslant k$ *with* $k-l$ *even. Suppose that* $p_{i,1}, ..., p_{i,l} \in \mathscr{T}_l$ *are such that* (3.7) *holds for each* $1 \leqslant i \leqslant n$. *Then*

$$Qf := \sum_{i=1}^{n} \sum_{j=1}^{l} \mathscr{B}[p_{i,j}](t_{i+1}, ..., t_{i+k-1}) \, \lambda_{i,j} f \, T_i^k \tag{3.10}$$

is the unique quasi-interpolant of the form (3.3) *which reproduces* $\mathscr{T}_l$. *Here* $\mathscr{B}$ *is the blossoming operator introduced in Theorem* 5.1.

Proof. Theorem 5.2 asserts that

$$p_{i,j} = \sum_{\nu=1}^{n} \mathscr{B}[p_{i,j}](t_{\nu+1}, ..., t_{\nu+k-1}) \, T_\nu^k,$$

and the result follows from Lemma 3.3. ∎

The operator in Theorem 3.4 can also be written as

$$Qf := \sum_{i=1}^{n} \mathscr{B}[V_{l,i} f](t_{i+1}, ..., t_{i+k-1})\, T_i^k, \tag{3.11}$$

where

$$V_{l,i} f := \sum_{j=1}^{l} (\lambda_{i,j} f)\, p_{i,j} \tag{3.12}$$

is the unique trigonometric polynomial in $\mathscr{T}_l$ which interpolates f in the sense that

$$\lambda_{i,j} V_{l,i} f = \lambda_{i,j} f, \qquad j = 1, ..., l. \tag{3.13}$$

Indeed, by the linearity of the blossom,

$$\mathscr{B}[V_{l,i} f](t_{i+1}, ..., t_{i+k-1}) = \sum_{j=1}^{l} (\lambda_{i,j} f)\, \mathscr{B}[p_{i,j}](t_{i+1}, ..., t_{i+k-1}).$$

We now give conditions under which Q reproduces the whole spline space $\mathscr{S}$. Recall that the *support of a linear functional* γ is the smallest interval $[c, d]$ such that if f vanishes on $[c, d]$, then $\gamma f = 0$.

THEOREM 3.5. *Let Q be a quasi-interpolant of the form* (3.3) *with* $l = k$ *which reproduces* $\mathscr{T}_k$, *and suppose that for each* $1 \leqslant i \leqslant n$, *there is a subinterval* $I_i := [t_{m_i}, t_{m_i+1}) \subset [t_i, t_{i+k}) \cap J$ *which contains the support of the functionals* $\lambda_{i,1}, ..., \lambda_{i,k}$. *Then Q is a linear projection onto the spline space* $\mathscr{S}$.

Proof. To show that Q is a projection, it suffices to prove that

$$\lambda_i T_\nu^k = \delta_{i,\nu}, \qquad \text{all} \quad i, \nu = 1, ..., n. \tag{3.14}$$

Fix $1 \leqslant i \leqslant n$ and consider the trigonometric polynomials $p_\nu := T_\nu^k|_{I_i}$ for $m_i + 1 - k \leqslant \nu \leqslant m_i$. The coefficients of T_μ^k in the trigonometric B-spline expansion of p_ν are $b_{\nu,\mu} = \delta_{\nu,\mu}$ for $m_i + 1 - k \leqslant \mu \leqslant m_i$. Thus, by (3.6),

$$\lambda_i T_\nu^k = \lambda_i p_\nu = b_{\nu,i} = \delta_{\nu,i}, \qquad \nu = m_i + 1 - k, ..., m_i.$$

This statement includes the fact that $\lambda_i T_i^k = 1$ since $m_i + 1 - k \leqslant i \leqslant m_i$. To complete the proof of (3.14), we note that by hypothesis, if $\nu < m_i + 1 - k$ or $\nu > m_i$, then the supports of λ_i and T_ν^k do not intersect, and so $\lambda_i T_\nu^k = 0$ for these values of ν. ∎

4. ERROR BOUNDS

Our goal in this section is to develop a general approach to obtaining error bounds for quasi-interpolants based on trigonometric splines. The key tool is the trigonometric Taylor expansion.

It is well known [9, 10] that $\mathscr{T}_k$ is the null space of the differential operator

$$L_k := \begin{cases} D(D^2+1)(D^2+4)\cdots\left(D^2+\left(\frac{k-1}{2}\right)^2\right), & k \text{ odd} \\ \left(D^2+\frac{1}{4}\right)\left(D^2+\frac{9}{4}\right)\cdots\left(D^2+\left(\frac{k-1}{2}\right)^2\right), & k \text{ even}, \end{cases} \tag{4.1}$$

where $L_0 := I$ and $L_1 := D$. For later use, we now introduce some related differential operators. Let $D_{k,0} = I$, and

$$D_{k,2j} := \left(D^2+\left(\frac{k+1-2j}{2}\right)^2\right)\cdots\left(D^2+\left(\frac{k-3}{2}\right)^2\right)\left(D^2+\left(\frac{k-1}{2}\right)^2\right), \tag{4.2}$$

for $1 \leqslant 2j \leqslant k$, and

$$D_{k,2j+1} := D_{k,2j} D, \qquad 1 \leqslant 2j+1 \leqslant k. \tag{4.3}$$

In addition, let $M_{k,0}$ be the identity operator, and let

$$M_{k,j} = \begin{cases} L_j, & \text{if } k-j \text{ is even}, \\ L_{j-1} D, & \text{if } k-j \text{ is odd}, \end{cases} \tag{4.4}$$

for $j > 0$. Note that all of the operators introduced here are constant coefficient differential operators, and the coefficient of the highest power of D is always 1. Their orders are indicated by their second subscript.

For later use we observe that

$$D_{\sigma,\sigma-j} s(x-t)^{\sigma-1} = 2^{j-\sigma} \frac{(\sigma-1)!}{(j-1)!} \begin{cases} s(x-t)^{j-1}, & \text{if } \sigma-j \text{ is even}, \\ c(x-t)\, s(x-t)^{j-1}, & \text{if } \sigma-j \text{ is odd}, \end{cases} \tag{4.5}$$

where $D_{\sigma,\sigma-j}$ operates on the x-variable.

We now present two types of trigonometric Taylor series.

LEMMA 4.1. *Let $\sigma \geqslant 1$, and let $f \in L_1^\sigma[J]$ for some interval J. Then for any point $t \in J$,*

$$f(x) = U_{\sigma, t} f(x) + R_{\sigma, t} f(x), \qquad x \in J, \tag{4.6}$$

where

$$U_{\sigma, t} f(x) := \frac{2^{\sigma-1}}{(\sigma-1)!} \sum_{j=1}^{\sigma} D_{\sigma, \sigma-j}[s(x-t)^{\sigma-1}]\, M_{\sigma, j-1} f(t) \tag{4.7}$$

is a trigonometric polynomial of order σ, and

$$R_{\sigma, t} f(x) = \frac{2^{\sigma-1}}{(\sigma-1)!} \int_t^x s(x-y)^{\sigma-1}\, L_\sigma f(y)\, dy. \tag{4.8}$$

Here $D_{\sigma, \sigma-j}$ operates on the x-variable.

Proof. The result follows directly by integration by parts. ∎

$U_{\sigma, t}$ is called a *trigonometric Taylor expansion of f about the point t*. In [9] it was defined recursively. Of course (4.7) could be written in terms of ordinary derivatives of f at the point t, but then the corresponding coefficients would not be in a form where (4.5) can be applied. The following lemma collects several useful facts about $U_{\sigma, t} f$.

LEMMA 4.2. *$U_{\sigma, t} f$ is a linear projection onto $\mathscr{T}_\sigma$. Moreover,*

$$M_{\sigma, v-1} U_{\sigma, t} f(t) = M_{\sigma, v-1} f(t), \qquad v = 1, ..., \sigma \tag{4.9}$$

and

$$D_{\sigma, v-1} U_{\sigma, t} f(t) = D_{\sigma, v-1} f(t), \qquad v = 1, ..., \sigma, \tag{4.10}$$

for all $f \in L_1^\sigma[J]$.

Proof. For fixed t, $U_{\sigma, t}$ is clearly a linear operator mapping functions $f \in L_1^\sigma[J]$ into $\operatorname{span}\{u_j\} \subseteq \mathscr{T}_\sigma$, where $u_j(x) := D_{\sigma, \sigma-j}[s(x-t)^{\sigma-1}]$, $j = 1, ..., \sigma$. It follows immediately from (4.8) that $U_{\sigma, t} f = f$ for all $f \in \mathscr{T}_\sigma$, and we conclude that $u_1, ..., u_\sigma$ must span all of $\mathscr{T}_\sigma$, and thus are a basis for it. Thus,

$$u_j(x) = U_{\sigma, t} u_j(x) = \frac{2^{\sigma-1}}{(\sigma-1)!} \sum_{j=1}^{\sigma} D_{\sigma, \sigma-j}[s(x-t)^{\sigma-1}]\, M_{\sigma, j-1} u_j(t)$$

for $j=1, ..., \sigma$ implies that

$$\frac{2^{\sigma-1}}{(\sigma-1)!} M_{\sigma, \nu-1} D_{\sigma, \sigma-j}[s(x-t)^{\sigma-1}]\,|_{x=t} = \delta_{\nu, j}, \qquad \nu, j=1, ..., \sigma. \tag{4.11}$$

Now (4.9) follows by applying $M_{\sigma, \nu-1}$ to (4.7) and evaluating at $x=t$.

Since the operators $M_{\sigma, \nu-1}$ have the form $D^{\nu-1}$ + lower-order terms, it follows that the analog of (4.9) holds with any set of derivative operators with the same property, and thus in particular for the operators $I, D, ..., D^{\sigma-1}$ and also $D_{\sigma, 0}, ..., D_{\sigma, \sigma-1}$. ∎

The Taylor expansion (4.7) produces a trigonometric polynomial in the space $\mathcal{T}_\sigma$. The following alternative version produces a trigonometric polynomial in the space $\mathcal{T}_{\sigma+1}$.

Lemma 4.3. *Let $\sigma \geqslant 1$, and let $f \in L_1^\sigma[J]$ for some interval J. Then for any point $t \in J$,*

$$f(x) = \tilde{U}_{\sigma, t} f(x) + \tilde{R}_{\sigma, t} f(x), \qquad x \in J, \tag{4.12}$$

where

$$\tilde{U}_{\sigma, t} f(x) := \frac{2^\sigma}{\sigma!} \sum_{j=1}^{\sigma} D_{\sigma+1, \sigma-j+1}[s(x-t)^\sigma]\, M_{\sigma+1, j-1} f(t) \tag{4.13}$$

and

$$\tilde{R}_{\sigma, t} f(x) := \frac{2^{\sigma-1}}{(\sigma-1)!} \int_t^x s(x-y)^{\sigma-1} \left[c(x-y)\, D + \frac{\sigma}{2} s(x-y) \right] L_{\sigma-1} f(y)\, dy. \tag{4.14}$$

Here $D_{\sigma+1, \sigma-j+1}$ operates on the x-variable.

Proof. The result follows by integration by parts. ∎

It is easy to see that $\tilde{U}_{\sigma, t} f$ satisfies the same interpolation conditions (4.9)–(4.10) as $U_{\sigma, t}$. The following lemma provides a general approach to obtaining error bounds for quasi-interpolants which reproduce the space of trigonometric polynomials $\mathcal{T}_l$.

Lemma 4.4. *Suppose the quasi-interpolant Q satisfies* (3.2), *where $k-l$ is even. Let m be such that $t_m \leqslant t \leqslant t_{m+1}$, and let $0 \leqslant r < \sigma \leqslant l$. Then for all $f \in L_\infty^\sigma[a, b]$,*

$$|D_{k, r}(f - Qf)(t)| \leqslant \sum_{i=m-k+1}^{m} |\lambda_i R|\, |D_{k, r} T_i^k(t)|, \tag{4.15}$$

where R is the remainder in the trigonometric Taylor expansion of order σ *about the point t as given in* (4.8) *if* $l-\sigma$ *is even, and in* (4.14) *if* $l-\sigma$ *is odd.*

Proof. We examine the case where $l-\sigma$ is even. Let $g := U_{\sigma,t} f$ be the trigonometric Taylor polynomial (4.7), and let R be the corresponding remainder term (4.8). Then since $Qg = g$ and $D_{k,r} R(t) = 0$,

$$\begin{aligned}|D_{k,r}(f-Qf)(t)| &= |D_{k,r}(f-g)(t) + D_{k,r}(Qg-Qf)(t)| \\ &\leqslant |D_{k,r} R(t)| + |D_{k,r} QR(t)| = |D_{k,r} QR(t)|.\end{aligned}$$

This completes the proof for $l-\sigma$ even. The proof of the odd case is similar using $g = \tilde{U}_{\sigma,t} f$. ∎

We have chosen to estimate the derivatives $D_{k,r}$ instead of the usual derivatives since in order to apply the lemma, we have to find bounds for the corresponding derivatives of the trigonometric B-splines. This is much easier if we use $D_{k,r}$ than if we use D^r.

LEMMA 4.5. *For* $k \geqslant 1$ *and* $0 \leqslant r \leqslant k-1$,

$$D_{k,r} T_i^k(x) = \frac{(k-1)!}{2^r (k-r-1)!} \sum_{\mu=0}^{r} (-1)^\mu \gamma_{i,r,\mu}^k(x)\, T_{i+\mu}^{k-r}(x), \tag{4.16}$$

where $\gamma_{i,0,v}^k = \delta_{v,0}$ *for all integers v, and where*

$$\gamma_{i,2\ell+1,\mu}^k(x) := \frac{c(x-t_{i+\mu})\, \gamma_{i,2\ell,\mu}^k + c(t_{i+\mu+k-2\ell-1}-x)\, \gamma_{i,2\ell,\mu-1}^k}{s(t_{i+\mu+k-2\ell-1}-t_{i+\mu})}, \tag{4.17}$$

$$\gamma_{i,2\ell+2,\mu}^k(x) := a_{i+\mu}^{k-2\ell} \gamma_{i,2\ell,\mu}^k + b_{i+\mu}^{k-2\ell} \gamma_{i,2\ell,\mu-1}^k + c_{i+\mu}^{k-2\ell} \gamma_{i,2\ell,\mu-2}^k. \tag{4.18}$$

Here

$$a_i^k := \frac{1}{s(t_{i+k-1}-t_i)\, s(t_{i+k-2}-t_i)}, \tag{4.19}$$

$$b_{i+1}^k := \frac{s(t_{i+k-1}+t_{i+k}-t_i-t_{i+1})}{s(t_{i+k-1}-t_i)\, s(t_{i+k}-t_{i+1})\, s(t_{i+k-1}-t_{i+1})}, \tag{4.20}$$

$$c_{i+2}^k := \frac{1}{s(t_{i+k}-t_{i+1})\, s(t_{i+k}-t_{i+2})}. \tag{4.21}$$

Proof. We recall [9, 10]

$$D_{k,1} T_i^k(x) = \left(\frac{k-1}{2}\right) \left[\frac{c(x-t_i)}{s(t_{i+k-1}-t_i)}\, T_i^{k-1}(x) - \frac{c(t_{i+k}-x)}{s(t_{i+k}-t_{i+1})}\, T_{i+1}^{k-1}(x) \right] \tag{4.22}$$

and

$$D_{k,2} T_i^k(x) = \frac{(k-1)(k-2)}{4} \left[a_i^k T_i^{k-2}(x) - b_{i+1}^k T_{i+1}^{k-2}(x) + c_{i+2}^k T_{i+2}^{k-2}(x) \right]. \tag{4.23}$$

We now proceed by induction. Suppose the formula (4.16) holds for $r = 2l$. Then the formula with $r = 2l+1$ follows by applying (4.22) and rearranging terms. Similarly, the result for $r = 2l+2$ can be established using (4.23). ∎

Given $1 \leqslant i \leqslant n$ and $i \leqslant m \leqslant i+k-1$, let

$$\underline{\Delta}_{i,m,j} := \min_{i \leqslant v \leqslant m < m+1 \leqslant v+j \leqslant i+k} (t_{v+j} - t_v), \tag{4.24}$$

$$\bar{\Delta}_{i,m,j} := \max_{i \leqslant v \leqslant m < m+1 \leqslant v+j \leqslant i+k} (t_{v+j} - t_v), \tag{4.25}$$

for $j = 1, ..., k-1$.

LEMMA 4.6. *Let* $1 \leqslant i \leqslant n$ *and* $0 \leqslant r \leqslant k-1$ *with* $k \geqslant 1$. *Then for all* $x \in [t_m, t_{m+1}) \subset [t_i, t_{i+k}]$,

$$|D_{k,r} T_i^k(x)| \leqslant \frac{2^{-r} C_{m,k,r,i,\Delta}}{s(\underline{\Delta}_{i,m,k-1}/2) \cdots s(\underline{\Delta}_{i,m,k-r}/2)}, \tag{4.26}$$

where

$$C_{m,k,r,i,\Delta} = \frac{(k-1)!}{(k-r-1)!\, c(\bar{\Delta}_{i,m,1}/2) \cdots c(\bar{\Delta}_{i,m,k-1}/2)}. \tag{4.27}$$

If $t_{i+k-1} - t_i \leqslant \pi$, *then*

$$C_{m,k,r,i,\Delta} \leqslant \frac{2^{(k-1)/2}(k-1)!}{(k-r-1)!}. \tag{4.28}$$

Proof. For $i \leqslant v \leqslant m < m+1 \leqslant v+j \leqslant i+k$ we have

$$s(t_{v+j} - t_j) = 2s((t_{v+j} - t_j)/2)\, c((t_{v+j} - t_j)/2) \geqslant 2s(\underline{\Delta}_{i,m,v}/2)\, c(\bar{\Delta}_{i,m,v}/2).$$

This follows since $(t_{v+j} - t_j)/2 < \pi$. It can then be shown that the $\gamma_{i,r,\mu}^k$ in Lemma 4.5 satisfy

$$0 < \gamma_{i,r,\mu}^k$$

$$\leqslant \frac{2^{-r} \binom{r}{\mu}}{s(\underline{\Delta}_{i,m,k-1}/2) \cdots s(\underline{\Delta}_{i,m,k-r}/2)\, c(\bar{\Delta}_{i,m,k-1}/2) \cdots c(\bar{\Delta}_{i,m,k-r}/2)}.$$

It was shown in [4] that

$$|T_i^{k-r}(x)| \leqslant \frac{1}{c(\bar{\Delta}_{i,m,k-r-1}/2) \cdots c(\bar{\Delta}_{i,m,1}/2)}, \qquad t_m \leqslant x < t_{m+1}. \tag{4.29}$$

Combining these two facts with (4.16) leads to (4.26). To establish (4.28), we observe that if $t_{i+k-1} - t_i \leqslant \pi$, then

$$c(\bar{\Delta}_{i,m,j}/2) \geqslant \cos(\pi/4) = 2^{-1/2}, \qquad j = 1, ..., k-1. \quad \blacksquare \tag{4.30}$$

5. BLOSSOMING AND TRIGONOMETRIC MARSDEN IDENTITIES

Our aim in this section is to find trigonometric B-spline expansions of arbitrary trigonometric polynomials $f \in \mathcal{T}_k$. Our starting point is the well-known *trigonometric Marsden identity* [9]

$$[s(y-x)]^{k-1} = \sum_{i=1}^{n} \Psi_{k,i}(y)\, T_i^k(x), \qquad x, y \in J, \tag{5.1}$$

where

$$\Psi_{k,i}(y) := \prod_{v=1}^{k-1} s(y - t_{i+v}). \tag{5.2}$$

Given $1 \leqslant l \leqslant k$, we now apply $D_{k,k-l}$. Then by (4.5), we get the expansion

$$s(y-x)^{l-1} = 2^{k-l} \frac{(l-1)!}{(k-1)!} \sum_{\mu=1}^{n} [D_{k,k-l} \Psi_{k,\mu}(y)]\, T_\mu^k(x), \qquad k-l \text{ even}, \tag{5.3}$$

and

$$c(y-x)\, s(y-x)^{l-1} = 2^{k-l} \frac{(l-1)!}{(k-1)!} \sum_{\mu=1}^{n} [D_{k,k-l} \Psi_{k,\mu}(y)]\, T_\mu^k(x), \qquad k-l \text{ odd}. \tag{5.4}$$

To derive more general Marsden-type identities, we make use of the concept of the blossom of a trigonometric polynomial, see [2].

THEOREM 5.1. *Fix integers l, k with $1 \leqslant l \leqslant k$ and $k-l$ even. For every $f \in \mathcal{T}_l$ and any $x_1, ..., x_{k-1}$ there exists a unique real-valued function $\mathscr{B}[f](x_1, ..., x_{k-1})$, called the blossom of f, which satisfies the following properties:*

(a) $\mathscr{B}[f]$ *is a symmetric function of the variables* $x_1, ..., x_{k-1}$,

(b) $\mathscr{B}[f]$ *is equal to f on the diagonal; i.e.,* $\mathscr{B}(f)(x, ..., x) = f(x)$, *for all* $x \in \mathbb{R}$,

(c) $\mathscr{B}[f](..., x_j, ...) \in \mathscr{T}_2$ *for all* $j = 1, ..., k-1$.

We can now compute the trigonometric B-spline expansion of an arbitrary trigonometric polynomial.

THEOREM 5.2. *For any* $f \in \mathscr{T}_k$,

$$f = \sum_{i=1}^{n} \mathscr{B}[f](t_{i+1}, ..., t_{i+k-1})\, T_i^k. \tag{5.5}$$

Proof. It is easy to check that

$$\mathscr{B}[s(y-x)^{k-1}](t_{i+1}, ..., t_{i+k-1}) = \Psi_{k,i}(y),$$

which implies that

$$s(y-x)^{k-1} = \sum_{i=1}^{n} \mathscr{B}[s(y-x)^{k-1}](t_{i+1}, ..., t_{i+k-1})\, T_i^k(x). \tag{5.6}$$

Now applying the derivative operator $D_{k,k-j}$ to both sides with respect to the y-variable and using the fact that it commutes with the blossoming operator $\mathscr{B}$ (operating on the x-variable), we get

$$D_{k,k-j} s(y-x)^{k-1} = \sum_{i=1}^{n} \mathscr{B}[D_{k,k-j} s(y-x)^{k-1}](t_{i+1}, ..., t_{i+k-1})\, T_i^k(x).$$

Setting $y = 0$, it follows from (5.3)–(5.4) that (5.5) holds for each of the polynomials $D_{k,k-j} s(x)^{k-1}$, $j = 1, ..., k$. Since these polynomials form a basis for $\mathscr{T}_k$, the linearity of the blossoming operator $\mathscr{B}$ implies that (5.5) holds for all $f \in \mathscr{T}_k$. ∎

We conclude this section by computing the blossom of a product of sine functions. In order to state the formula, it will be convenient to introduce the following notation for multiple sums. Suppose $A_\imath := A_{i_1, ..., i_m}$ are real numbers defined for all integers $1 \leqslant i_1, ..., i_m \leqslant k$. Then we define

$$\sum_{\imath=1}^{k}{}_m A_\imath := \sum_{i_1=1}^{k} \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^{k} \cdots \sum_{\substack{i_m=1 \\ i_m \neq i_1, i_2, ..., i_{m-1}}}^{k} A_{i_1, ..., i_m}, \tag{5.7}$$

where $\imath$ stands for the multi-index $(i_1, ..., i_m)$.

LEMMA 5.3. *Fix integers l, k with $1 \leqslant l \leqslant k$ and $k-l$ even. Then for any $\theta_1, ..., \theta_{l-1}$ and $x_1, ..., x_{k-1}$,*

$$\mathscr{B}\left[\prod_{v=1}^{l-1} s(\cdot - \theta_v)\right](x_1, ..., x_{k-1})$$
$$= \frac{1}{(k-1)!} \sum_{\imath=1}^{k-1}{}_{k-1} \prod_{v=1}^{l-1} s(x_{i_v} - \theta_v) \prod_{v=1}^{(k-l)/2} c(x_{i_{l+2v-1}} - x_{i_{l+2v-2}}), \tag{5.8}$$

and

$$\mathscr{B}\left[c(\cdot - \theta_1) \prod_{v=2}^{l-1} s(\cdot - \theta_v)\right](x_1, ..., x_{k-1})$$
$$= \frac{1}{(k-1)!} \sum_{\imath=1}^{k-1}{}_{k-1} c(x_{i_1} - \theta_1) \prod_{v=2}^{l-1} s(x_{i_v} - \theta_v) \prod_{v=1}^{(k-l)/2} c(x_{i_{l+2v-1}} - x_{i_{l+2v-2}}). \tag{5.9}$$

Proof. The sum is over all permutations $i_1, ..., i_{k-1}$ of the integers $1, ..., k-1$. Clearly, the right-hand side of (5.8) is symmetric with respect to $x_1, ..., x_{k-1}$. Moreover, it has the diagonal property (b) of Theorem 5.1, since if we set $x_1 = \cdots = x_{k-1} = x$, the sum involves exactly $(k-1)!$ copies of the same product. Since (c) is also satisfied, the result follows. Equation (5.9) follows from (5.8) by differentiating both sides with respect to θ_1. ∎

6. QUASI-INTERPOLANTS BASED ON DERIVATIVES

Fix k, and suppose the $T_i^k(x)$ are the trigonometric B-splines associated with an extended knot sequence (2.1). Let $t_i \leqslant \tau_i \leqslant t_{i+k}$, for $i = 1, ..., n$. In this section we examine trigonometric spline quasi-interpolants which are based on sampling a function and its derivatives at the τ_i. Given $1 \leqslant l \leqslant k$, let

$$Q_{k,l}^D f := \sum_{i=1}^{n} (\lambda_{l,i}^D f)\, T_i^k, \tag{6.1}$$

where

$$\lambda_{l,i}^D := \frac{2^{k-1}}{(k-1)!} \sum_{j=1}^{l} (-1)^{j-1} D_{k,k-j} \Psi_{k,i}(\tau_i)\, M_{k,j-1} f(\tau_i). \tag{6.2}$$

Here $\Psi_{k,i}$ are the functions (5.2) appearing in Marsden's identity, and $M_{k,j-1}$ are the operators defined in (4.4). The superscript on $Q_{k,l}^D$ is meant to remind us that these quasi-interpolants are based on derivatives.

Clearly, $Q^D_{k,l}$ is a linear operator whose domain includes all functions which are piecewise C^{l-1} on each of the subintervals defined by the partition Δ, and whose range is contained in the trigonometric spline space $\mathscr{S}$. In particular, we may take either left or right derivatives whenever necessary. The operators $Q^D_{k,l}$ are analogs of the classical de Boor–Fix quasi-interpolant based on polynomial splines [1] and the evaluation of f and its (ordinary) derivatives at each sample point. We now show that $Q^D_{k,l}$ reproduces $\mathscr{T}_l$ provided $k-l$ is even.

THEOREM 6.1. *Suppose $1 \leqslant l \leqslant k$, and that $k-l$ is even. Then $Q^D_{k,l} f = f$ for all $f \in \mathscr{T}_l$.*

Proof. We apply Theorem 3.4 with $\lambda_{i,j} f = M_{k,j-1} f(\tau_i)$ and the trigonometric polynomials $p_{i,j}(x) = ((-1)^{j-1} 2^{k-1}/(k-1)!)\, D_{k,k-j} [s(x-\tau_i)^{k-1}]$ which appear in the Taylor expansion (4.7) of order l about the point τ_i. It follows from (4.11) with $\sigma = k$ that these functionals and polynomials satisfy (3.7). Finally, (5.3)–(5.4) imply that (3.10) can be rewritten in the form (6.1)–(6.2). ∎

THEOREM 6.2. *The operator $Q^D_{k,k}$ is a linear projection onto the spline space spanned by the $\{T^k_i\}_{i=1}^n$.*

Proof. Since $Q^D_{k,l}$ reproduces $\mathscr{T}_k$ and the supports of the $\lambda_{i,1}, ..., \lambda_{i,k}$ are clearly all in one knot interval I_i for each $i = 1, ..., n$, the result follows immediately from Theorem 3.5. ∎

We now give a few examples with different choices of k, l, and the τ_i:

$$Q^D_{1,1} f := \sum_{i=1}^{n} f(\tau_i)\, T^1_i, \tag{6.3}$$

$$Q^D_{2,2} f := \sum_{i=1}^{n} [c(\tau_i - t_{i+1})\, f(\tau_i) - 2s(\tau_i - t_{i+1})\, Df(\tau_i)]\, T^2_i, \tag{6.4}$$

$$\tilde{Q}^D_{2,2} f := \sum_{i=1}^{n} f(t_{i+1})\, T^2_i, \tag{6.5}$$

$$Q^D_{3,1} f := \sum_{i=1}^{n} c(t_{i+2} - t_{i+1})\, f(\tau_i)\, T^3_i, \tag{6.6}$$

$$\begin{aligned} Q^D_{3,3} f := \sum_{i=1}^{n} [&c(t_{i+2} - t_{i+1})\, f(\tau_i) - s(2\tau_i - t_{i+1} - t_{i+2})\, Df(\tau_i) \\ &+ 2s(\tau_i - t_{i+1})\, s(\tau_i - t_{i+2})\, D^2 f(\tau_i)]\, T^3_i, \end{aligned} \tag{6.7}$$

$$\tilde{Q}^D_{3,3} f := \sum_{i=1}^{n} \left[c(t_{i+2} - t_{i+1})\, f\left(\frac{t_{i+2} + t_{i+2}}{2}\right) \right.$$
$$\left. - 2s\left(\frac{t_{i+2} - t_{i+1}}{2}\right)^2 D^2 f\left(\frac{t_{i+1} + t_{i+2}}{2}\right)\right] T^3_i. \tag{6.8}$$

Except for $Q^D_{3,1}$, all of these quasi-interpolants are projections with range in their associated spline spaces, while $Q^D_{3,1}$ only reproduces $\mathscr{T}_1$. The quasi-interpolant $\tilde{Q}^D_{2,2}$ is obtained from $Q^D_{2,2}$ by choosing $\tau_i = t_{i+1}$ for all i, and $\tilde{Q}^D_{3,3}$ is obtained from $Q^D_{3,3}$ by choosing $\tau_i = (t_{i+1} + t_{i+2})/2$ for all $1 \leqslant i \leqslant n$.

We conclude this section by stating a result on how well the quasi-interpolant $Q^D_{k,l} f$ approximates a smooth function f. Our error bounds depend on the "mesh size"

$$\bar{\Delta} := \max_{k \leqslant i \leqslant n} (t_{i+1} - t_i). \tag{6.9}$$

We recall that the interval on which our quasi-interpolants are defined is $J := [t_k, t_{n+1}]$.

THEOREM 6.3. *Let $1 \leqslant \sigma \leqslant l \leqslant k$ with $k - l$ even, and fix $1 \leqslant p \leqslant q \leqslant \infty$. If $l - \sigma$ is even, then there exists a constant $K = K_{k,r,\sigma,\Delta}$ such that*

$$\|D_{k,r}(f - Q^D_{k,l} f)\|_{L_q[J]} \leqslant K \bar{\Delta}^{\sigma - r + (1/q) - (1/p)} \|L_\sigma f\|_{L_p[J]} \tag{6.10}$$

for all $0 \leqslant r < \sigma$ and all $f \in L^\sigma_p[J]$. If $l - \sigma$ is odd, then

$$\|D_{k,r}(f - Q^D_{k,l} f)\|_{L_q[J]}$$
$$\leqslant K \bar{\Delta}^{\sigma - r + (1/q) - (1/p)} \left[\|DL_{\sigma-1} f\|_{L_p[J]} + \frac{k\sigma}{2} s(\bar{\Delta}) \|L_{\sigma-1} f\|_{L_p[J]} \right] \tag{6.11}$$

for all $0 \leqslant r < \sigma$ and all $f \in L^\sigma_p[J]$.

The proof of this theorem is contained in Section 8, where we give a local version of the theorem, and an explicit formula for the constant K. In Section 10 we discuss conditions under which the constant is mesh-independent.

7. QUASI-INTERPOLANTS BASED ON POINT EVALUATORS

In this section we construct quasi-interpolants based on point evaluators. Given $1 \leqslant l \leqslant k$, let

$$t_i \leqslant \tau_{i,1} < \tau_{i,2} < \cdots < \tau_{i,l} \leqslant t_{i+k} \tag{7.1}$$

lie in the support $[t_i, t_{i+k}]$ of the B-spline T_i^k for $1 \leqslant i \leqslant n$. For each $1 \leqslant j \leqslant l$, let

$$\alpha_{i,j}^P := \frac{\sum_{\iota=1}^{k-1} {}_{k-1} \prod_{\nu=1}^{l-1} s(t_{i+i_\nu} - \theta_\nu) \prod_{\nu=1}^{(k-l)/2} c(t_{i+i_{l+2\nu-1}} - t_{i+i_{l+2\nu-2}})}{(k-1)! \prod_{\substack{\nu=1 \\ \nu \neq j}}^{l} s(\tau_{i,\nu} - \tau_{i,j})}, \tag{7.2}$$

where $\{\theta_1, ..., \theta_{l-1}\} := \{\tau_{i,1}, ..., \tau_{i,j-1}, \tau_{i,j+1}, ..., \tau_{i,l}\}$. The operator of interest in this section is

$$Q_{k,l}^P f := \sum_{i=1}^{n} (\lambda_{l,i}^P f)\, T_i^k, \tag{7.3}$$

with

$$\lambda_{l,i}^P f := \sum_{j=1}^{l} \alpha_{i,j}^P f(\tau_{i,j}). \tag{7.4}$$

Clearly, $Q_{k,l}^P$ is a linear operator mapping continuous functions on J into the spline space $\mathscr{S}$ spanned by the $\{T_i^k\}_{i=1}^n$. The superscript on $Q_{k,l}^P$ is meant to remind us that these quasi-interpolants are based on point evaluations of the function. We now show that $Q_{k,l}^P$ reproduces trigonometric polynomials of order l.

THEOREM 7.1. *Suppose $1 \leqslant l \leqslant k$ with $k-l$ even. Then $Q_{k,l}^P f = f$ for all $f \in \mathscr{T}_l$.*

Proof. It is easy to verify that the trigonometric polynomials

$$p_{i,j}(x) := \frac{\prod_{\substack{\nu=1 \\ \nu \neq j}}^{l} s(x - \tau_{i,\nu})}{\prod_{\substack{\nu=1 \\ \nu \neq j}}^{l} s(\tau_{i,j} - \tau_{i,\nu})} \tag{7.5}$$

satisfy

$$p_{i,j}(\tau_{i,\nu}) = \delta_{\nu,j}, \qquad \nu, j = 1, ..., l.$$

The formula in Theorem 5.3 implies that $\alpha_{i,j}^P = \mathscr{B}[p_{i,j}](t_{i+1}, ..., t_{i+k-1})$, and the result follows from Theorem 3.4. ∎

Theorem 3.5 can now be applied to give conditions under which $Q^P_{k,k}$ is a projection onto the spline space $\mathscr{S}$.

THEOREM 7.2. *Suppose that for each* $1 \leqslant i \leqslant n$, *there exists an integer* m_i *with* $t_i \leqslant t_{m_i} \leqslant \tau_{i,1} < \tau_{i,2} < \cdots < \tau_{i,k} \leqslant t_{m_i+1} \leqslant t_{i+k}$. *Then the quasi-interpolant* $Q^P_{k,k}$ *is a linear projection onto the spline space* $\mathscr{S}$.

We now give a few examples for various choice of k, l and the sample points $\tau_{i,j}$:

$$Q^P_{1,1} f := \sum_{i=1}^{n} f(\tau_i)\, T^1_i, \tag{7.6}$$

$$Q^P_{2,2} f := \sum_{i=1}^{n} \left[\frac{s(\tau_{i,2} - t_{i+1})}{s(\tau_{i,2} - \tau_{i,1})} f(\tau_{i,1}) + \frac{s(\tau_{i,1} - t_{i+1})}{s(\tau_{i,1} - \tau_{i,2})} f(\tau_{i,2}) \right] T^2_i, \tag{7.7}$$

$$\tilde{Q}^P_{2,2} f := \sum_{i=1}^{n} f(t_{i+1})\, T^2_i, \tag{7.8}$$

$$Q^P_{3,1} f := \sum_{i=1}^{n} c(t_{i+2} - t_{i+1})\, f(\tau_{i,1})\, T^3_i, \tag{7.9}$$

$$\begin{aligned} Q^P_{3,3} f := \sum_{i=1}^{n} \Bigg[& \frac{\begin{pmatrix} s(\tau_{i,2} - t_{i+1})\, s(\tau_{i,3} - t_{i+2}) \\ \quad + s(\tau_{i,2} - t_{i+2})\, s(\tau_{i,3} - t_{i+1}) \end{pmatrix}}{2s(\tau_{i,2} - \tau_{i,1})\, s(\tau_{i,3} - \tau_{i,1})} f(\tau_{i,1}) \\ & + \frac{\begin{pmatrix} s(\tau_{i,1} - t_{i+1})\, s(\tau_{i,3} - t_{i+2}) \\ \quad + s(\tau_{i,1} - t_{i+2})\, s(\tau_{i,3} - t_{i+1}) \end{pmatrix}}{2s(\tau_{i,1} - \tau_{i,2})\, s(\tau_{i,3} - \tau_{i,2})} f(\tau_{i,2}) \\ & + \frac{\begin{pmatrix} s(\tau_{i,1} - t_{i+1})\, s(\tau_{i,2} - t_{i+2}) \\ \quad + s(\tau_{i,1} - t_{i+2})\, s(\tau_{i,2} - t_{i+1}) \end{pmatrix}}{2s(\tau_{i,1} - \tau_{i,3})\, s(\tau_{i,2} - \tau_{i,3})} f(\tau_{i,3}) \Bigg] T^3_i \end{aligned} \tag{7.10}$$

$$\begin{aligned} \tilde{Q}^P_{3,3} f := \sum_{i=1}^{n} \Bigg[& -\frac{1}{2} f(t_{i+1}) + 2c \left(\frac{t_{i+2} - t_{i+1}}{2} \right)^2 \\ & \times f\left(\frac{t_{i+1} + t_{i+2}}{2} \right) - \frac{1}{2} f(t_{i+2}) \Bigg] T^3_i. \end{aligned} \tag{7.11}$$

The operators $Q^P_{1,1}$, $\tilde{Q}^P_{2,2}$, and $\tilde{Q}^P_{3,3}$ are linear projections onto the associated spline spaces. The quasi-interpolants $Q^P_{2,2}$ and $Q^P_{3,3}$ reproduce $\mathscr{T}_2$ and $\mathscr{T}_3$, respectively. They are also linear projections provided that for each i, the $\tau_{i,j}$ are chosen in a single knot interval contained in the support of T^k_i. The quasi-interpolant $\tilde{Q}^P_{2,2}$ is obtained from $Q^P_{2,2}$ by choosing $\tau_{i,1} = t_{i+1}$ for all i, and $\tilde{Q}^P_{3,3}$ is obtained from $Q^P_{3,3}$ by choosing $\tau_{i,1} = t_{i+1}$,

$\tau_{i,2}=(t_{i+1}+t_{i+2})/2$, and $\tau_{i,3}=t_{i+2}$ for all i. A periodic version of the quasi-interpolant $\tilde{Q}^P_{3,3}$ was used in [11] for fitting data on the sphere.

We conclude this section by stating a result on how well the quasi-interpolant $Q^P_{k,l}f$ approximates smooth functions f in terms of the mesh size $\bar{\Delta}$ defined in (6.9).

THEOREM 7.3. *Let $1\leqslant\sigma\leqslant l\leqslant k$ with $k-l$ even, and fix $1\leqslant p\leqslant q\leqslant\infty$. Then if $l-\sigma$ is even, there exists a constant $K=K_{k,r,\sigma,\Delta}$ such that*

$$\|D_{k,r}(f-Q^P_{k,l}f)\|_{L_q[J]}\leqslant K\bar{\Delta}^{\sigma-r+(1/q)-(1/p)}\,\|L_\sigma f\|_{L_p[J]} \tag{7.12}$$

for all $f\in L^\sigma_p[J]$ and all $0\leqslant r<\sigma$. Similarly, if $l-\sigma$ is odd, then

$$\|D_{k,r}(f-Q^P_{k,l}f)\|_{L_q[J]}$$
$$\leqslant K\bar{\Delta}^{\sigma-r+(1/q)-(1/p)}\left[\,\|DL_{\sigma-1}f\|_{L_p[J]}+\frac{k\sigma}{2}\,s(\bar{\Delta})\,\|L_{\sigma-1}f\|_{L_p[J]}\right]. \tag{7.13}$$

The proof of this theorem is contained in Section 9, where we give a local version of the theorem, and an explicit formula for the constant K.

8. ERROR BOUNDS FOR $Q^D_{k,l}$

In this section we establish both local and global error bounds for the quasi-interpolant $Q^D_{k,l}$ defined in (6.1). We begin by finding explicit formulae for the values of the linear functionals $\lambda^D_{l,i}$ given in (6.2) when applied to the kernels of the remainders in the Taylor expansions (4.6) and (4.12).

LEMMA 8.1. *Suppose $1\leqslant\sigma\leqslant l\leqslant k$ with $k-l$ and $k-\sigma$ both even. Then*

$$\lambda^D_{l,i}s(\cdot-y)^{\sigma-1}=\frac{1}{(k-1)!}\sum_{\iota=1}^{k-1}{}_{k-1}A_\iota\prod_{\nu=1}^{\sigma-1}s(t_{i+i_\nu}-y), \tag{8.1}$$

and

$$\lambda^D_{l,i}c(\cdot-y)\,s(\cdot-y)^{\sigma-2}$$
$$=\frac{1}{(k-1)!}\sum_{\iota=1}^{k-1}{}_{k-1}A_\iota c(t_{i+i_1}-y)\prod_{\nu=2}^{\sigma-1}s(t_{i+i_\nu}-y), \tag{8.2}$$

for all $y \in J$ *and all* $1 \leqslant i \leqslant n$, *where*

$$A_\iota = \prod_{\nu=1}^{(k-\sigma)/2} c(t_{i+i_{\sigma+2\nu-1}} - t_{i+i_{\sigma+2\nu-2}}).$$

Proof. Fix $1 \leqslant i \leqslant n$. Since Q reproduces $\mathcal{T}_\sigma$, by Theorem 5.2,

$$\lambda_{l,i}^D f = \mathcal{B}[f](t_{i+1}, ..., t_{i+k-1})$$

for any $f \in \mathcal{T}_\sigma$. Thus, (8.1) and (8.2) follow by applying Lemma 5.3 to $f = s(\cdot - y)^{\sigma-1}$ and $f = c(\cdot - y)\, s(\cdot - y)^{\sigma-2}$, respectively. ∎

We are now ready to establish a local error bound. First we need some additional notation. Given $k \leqslant m \leqslant n$ and $1 \leqslant j \leqslant k$, define

$$\underline{\Delta}_{m,j} := \min_{m-j+1 \leqslant i \leqslant m} (t_{i+j} - t_i), \tag{8.3}$$

$$\overline{\Delta}_{m,j} := \max_{m-j+1 \leqslant i \leqslant m} (t_{i+j} - t_i), \tag{8.4}$$

$$\underline{\Delta}_j := \min_{k \leqslant m \leqslant n} \underline{\Delta}_{m,j}, \tag{8.5}$$

$$\overline{\Delta}_j := \max_{k \leqslant m \leqslant n} \overline{\Delta}_{m,j}. \tag{8.6}$$

Throughout the remainder of the paper we assume that

$$0 < t_{i+k-1} - t_i \leqslant \pi, \qquad i = 1, ..., n. \tag{8.7}$$

In estimating various factors involving the function $s(x) = \sin(x/2)$, we note that

$$\frac{x}{\pi} \leqslant s(x) \leqslant \frac{x}{2}, \qquad \text{if} \quad 0 \leqslant x \leqslant \pi. \tag{8.8}$$

THEOREM 8.2. *Let* $1 \leqslant \sigma \leqslant l \leqslant k$ *with* $k - l$ *even, and let* $1 \leqslant p, q \leqslant \infty$. *Suppose* m *is an integer such that* $a \leqslant t_m < t_{m+1} \leqslant b$, *and let* I_m *be the smallest closed interval containing* $[t_m, t_{m+1}]$ *and* $\{\tau_i\}_{i=m+1-k}^m$. *If* $l - \sigma$ *is even, then*

$$\|D_{k,r}(f - Q_{k,l}^D f)\|_{L_q[t_m, t_{m+1}]} \leqslant K_m \overline{\Delta}_{m,k}^{\sigma - r + (1/q) - (1/p)} \|L_\sigma f\|_{L_p[I_m]} \tag{8.9}$$

for all $f \in L_p^\sigma[I_m]$ *and all* $0 \leqslant r < \sigma$, *where*

$$K_m := K_{m,k,r,\sigma,\Delta} := \frac{k 2^r}{(\sigma-1)!} C_{m,k,r,\Delta} \prod_{\nu=1}^{r} \frac{s(\overline{\Delta}_{m,k-1}/2)}{s(\underline{\Delta}_{m,k-\nu}/2)},$$

where

$$C_{m,k,r,\Delta} = \frac{(k-1)!}{(k-r-1)!\; c(\bar{\Delta}_{m,1}/2) \cdots c(\bar{\Delta}_{m,k-1}/2)} \leqslant \frac{2^{(k-1)/2}(k-1)!}{(k-r-1)!}. \tag{8.10}$$

Similarly, if $l-\sigma$ is odd, then

$$\|D_{k,r}(f - Q^D_{k,l} f)\|_{L_q[t_m, t_{m+1}]} \leqslant K_m \bar{\Delta}_{m,k}^{\sigma - r + (1/q) - (1/p)} \left[\|DL_{\sigma-1} f\|_{L_p[I_m]} + \frac{\sigma}{2} s(\bar{\Delta}_{m,k-1}) \|L_{\sigma-1} f\|_{L_p[I_m]} \right]. \tag{8.11}$$

Proof. We apply Lemma 4.4 with $\lambda_i = \lambda^D_{l,i}$. Suppose $t_m \leqslant t < t_{m+1}$. By Lemma 4.6,

$$|D_{k,r}(f - Q^D_{k,l} f)(t)| \leqslant 2^{-r} C_{m,k,r,\Delta} \sum_{i=m-k+1}^{m} \frac{|\lambda_i R|}{s(\underline{\Delta}_{i,m,k-1}/2) \cdots s(\underline{\Delta}_{i,m,k-r}/2)}, \tag{8.12}$$

where R is the remainder in the trigonometric Taylor expansion of order σ about the point t as given in (4.8) if $l-\sigma$ is even, and in (4.14) if $l-\sigma$ is odd. Fix $m+1-k \leqslant i \leqslant m$, and let $J_{t,\tau_i} \subseteq I_m$ be the smallest interval containing both t and τ_i.

First we examine the case $l-\sigma$ even. By (4.8) we have

$$|\lambda_i R| \leqslant \frac{2^{\sigma-1}}{(\sigma-1)!} \int_t^{\tau_i} |\lambda_i(s(\cdot - y)^{\sigma-1})|\, |L_\sigma f(y)|\, dy$$

$$\leqslant \max_{y \in J_{t,\tau_i}} |\lambda_i(s(\cdot - y)^{\sigma-1})|\, \frac{2^{\sigma-1}}{(\sigma-1)!} \bar{\Delta}_{m,k}^{1-1/p} \|L_\sigma f\|_{L_p[I_m]}.$$

Now using $|\sin x| \leqslant 2\,|\sin x/2|$ and (8.8), (8.1) implies

$$|\lambda_i(s(\cdot - y)^{\sigma-1})| \leqslant \max_{1 \leqslant i_1 < \cdots < i_{\sigma-1} \leqslant k-1} \prod_{\nu=1}^{\sigma-1} |s(y - t_{i+i_\nu})|$$

$$\leqslant 2^{2r-\sigma+1} \max_{1 \leqslant i_1 < \cdots < i_r \leqslant k-1} \prod_{\nu=1}^{r} |s((y - t_{i+i_\nu})/2)|\, \bar{\Delta}_{m,k}^{\sigma-r-1}$$

$$\leqslant 2^{2r-\sigma+1} \prod_{\nu=1}^{r} s(\bar{\Delta}_{m,k-1}/2)\, \bar{\Delta}_{m,k}^{\sigma-r-1} \tag{8.13}$$

for $y \in J_{t,\tau_i}$. Thus (8.12) implies (8.9) for $q=\infty$. The result for general $1 \leqslant q < \infty$ follows by integrating the qth power over the interval $[t_m, t_{m+1}]$.

Now let $l-\sigma$ be odd. Then the remainder R is given by (4.14) and we have

$$|\lambda_i R| \leqslant \frac{2^{\sigma-1}}{(\sigma-1)!} \left[\int_t^{\tau_i} |\lambda_i(c(\cdot - y)\, s(\cdot - y)^{\sigma-1})|\; |DL_{\sigma-1} f(y)|\, dy \right.$$
$$\left. + \frac{\sigma}{2} \int_t^{\tau_i} |\lambda_i(s(\cdot - y)^{\sigma})|\; |L_{\sigma-1} f(y)|\, dy \right].$$

Since $k-l$ is even and $l-\sigma$ is odd, $k-\sigma-1$ is even. We can therefore use (8.1) and (8.2) with σ replaced by $\sigma+1$. This gives

$$|\lambda_i(s(\cdot - y)^{\sigma})| \leqslant 2^{2r-\sigma+1} \prod_{v=1}^{r} s(\bar{\Delta}_{m,k-1}/2)\, s(\bar{\Delta}_{m,k-1})\, \bar{\Delta}_{m,k}^{\sigma-r-1}$$

and

$$|\lambda_i c(\cdot - y)(s(\cdot - y)^{\sigma-1})|$$
$$\leqslant \max_{1 \leqslant i_1 < \cdots < i_{\sigma-1} \leqslant k-1} \prod_{v=1}^{\sigma-1} |s(y - t_{i+i_v})|$$
$$\leqslant 2^{2r-\sigma+1} \max_{1 \leqslant i_1 < \cdots < i_r \leqslant k-1} \prod_{v=1}^{r} |s((y - t_{i+i_v})/2)|\; \bar{\Delta}_{m,k}^{\sigma-r-1}$$
$$\leqslant 2^{2r-\sigma+1} \prod_{v=1}^{r} s(\bar{\Delta}_{m,k-1}/2)\, \bar{\Delta}_{m,k}^{\sigma-r-1} \tag{8.14}$$

for $y \in J_{t,\tau_i}$. Now (8.12) implies (8.11) for $q=\infty$. The result for general $1 \leqslant q < \infty$ follows by integrating the qth power over the interval $[t_m, t_{m+1}]$. ∎

We are now ready to prove Theorem 6.3. First we note that for all $1 \leqslant j \leqslant k$ and all m, $\bar{\Delta}_{m,j} \leqslant \bar{\Delta}_j \leqslant j\bar{\Delta}$ and thus $s(\bar{\Delta}_{m,j}) \leqslant s(\bar{\Delta}_j)$ since we are assuming $\bar{\Delta}_j \leqslant \pi$. Consider the case $l-\sigma$ even. Raising (8.9) to the qth power and summing over all v such that $a \leqslant t_{m_v} < t_{m_v+1} \leqslant b$, we have

$$\left(\sum_v \|D_{k,r}(f - Q_{k,l}^D f)\|_{L_q[t_{m_v}, t_{m_v+1}]}^q \right)^{1/q}$$
$$\leqslant (\max_{k \leqslant m \leqslant n} K_m)\, \bar{\Delta}_k^{\sigma-r+(1/q)-(1/p)} \left(\sum_v \|L_\sigma f\|_{L_p[I_{m_v}]}^q \right)^{1/q}.$$

But for $p \leqslant q$, Jensen's inequality (see [10]) yields

$$\left(\sum_\nu \|L_\sigma f\|^q_{L_p[I_{m_\nu}]}\right)^{1/q} \leqslant \left(\sum_\nu \|L_\sigma f\|^p_{L_p[I_{m_\nu}]}\right)^{1/p}$$
$$\leqslant (2k-1)\, \|L_\sigma f\|_{L_p[J]},$$

Since $I_{m_\nu} \subset [t_{m_\nu+1-k}, t_{m_\nu+k}]$, and thus any piece of J is added into the sum at most $(2k-1)$ times. This gives (6.10) with

$$K := \frac{k(2k-1)\, 2^r}{(\sigma-1)!}\, C_{k,r,\Delta} \prod_{\nu=1}^{r} \frac{|s(\bar{\Delta}_{k-1}/2)|}{|s(\underline{\Delta}_{k-\nu}/2)|}, \tag{8.15}$$

where

$$C_{k,r,\Delta} := \frac{(k-1)!}{(k-r-1)!\; c(\bar{\Delta}_{k-1}/2) \cdots c(\bar{\Delta}_1/2)}. \tag{8.16}$$

To establish the result for $l-\sigma$ odd, we repeat the proof, starting with (8.11). ∎

9. ERROR BOUNDS FOR $Q^P_{k,l}$

In this section we establish both local and global error bounds for the quasi-interpolant $Q^P_{k,l}$ defined in (7.3). Throughout the section we assume that (8.7) holds, and use the following notation:

$$\Theta_m := \min_{\substack{1 \leqslant j \leqslant l-1 \\ m+1-k \leqslant i \leqslant m}} (\tau_{i,j+1} - \tau_{i,j}),$$

$$\Theta := \min_{1 \leqslant m \leqslant n} \Theta_m.$$

THEOREM 9.1. *Let $1 \leqslant \sigma \leqslant l \leqslant k$ with $k-l$ even, and fix $1 \leqslant p, q \leqslant \infty$. Given $k \leqslant m \leqslant n$, let I_m be the smallest closed interval containing $[t_m, t_{m+1}]$ and $\{\tau_{i,j}\}_{i=m+1-k,\, j=1}^{m,\, l}$. If $l-\sigma$ is even, then*

$$\|D_{k,r}(f - Q^P_{k,l} f)\|_{L_q[t_m, t_{m+1}]} \leqslant K_m \bar{\Delta}_{m,k}^{\sigma-r+(1/q)-(1/p)}\, \|L_\sigma f\|_{L_p[I_m]} \tag{9.1}$$

for all $f \in L^\sigma_p[I_m]$ and all $0 \leqslant r < \sigma$, where

$$K_m := K_{m,k,r,l,\sigma,\Delta,\Theta_m}$$
$$:= \frac{kl\, 2^r}{(\sigma-1)!} \left(\frac{s(\bar{\Delta}_{m,k-1})}{s(\Theta_m)}\right)^{l-1} \prod_{\nu=1}^{r} \frac{s(\bar{\Delta}_{m,k}/2)}{s(\underline{\Delta}_{m,k-\nu}/2)}\, C_{m,k,r,\Delta},$$

where $C_{m,k,r,\Delta}$ *is given in* (8.10). *Similarly, if* $l-\sigma$ *is odd, then*

$$\|D_{k,r}(f-Q_{k,l}^{P}f)\|_{L_q[t_m,t_{m+1}]}$$
$$\leqslant K_m \bar{\Delta}_{m,k}^{\sigma-r+(1/q)-(1/p)}\left[\|DL_{\sigma-1}f\|_{L_p[I_m]}+\frac{\sigma}{2}s(\bar{\Delta}_{m,k})\|L_{\sigma-1}f\|_{L_p[I_m]}\right]. \tag{9.2}$$

Proof. By the definition of $Q_{k,l}^{P}$, we can apply Lemma 4.4 with

$$\lambda_i f=\lambda_{l,i}^{P}f:=\sum_{j=1}^{l}\alpha_{i,j}^{P}f(\tau_{i,j}),$$

where $\alpha_{i,j}^{P}$ is given by (7.2) for $j=1, ..., l$ and $i=1, ..., n$.

First we examine the case $l-\sigma$ even, and derive a pointwise estimate. Let $t_m\leqslant t<t_{m+1}$, and let R be the remainder (4.8) in the Taylor expansion (4.6) of f about the point t. Fix $m-k+1\leqslant i\leqslant m$. We need an estimate for

$$|\lambda_i R|\leqslant\sum_{j=1}^{l}|\alpha_{i,j}^{P}|\,|R(\tau_{i,j})|\leqslant l\max_{1\leqslant j\leqslant l}|\alpha_{i,j}^{P}R(\tau_{i,j})|. \tag{9.3}$$

By (7.2),

$$|\alpha_{i,j}^{P}|\leqslant\max_{1\leqslant i_1<\cdots<i_{l-1}\leqslant k-1}\frac{\prod_{v=1}^{l-1}|s(\theta_v-t_{i+i_v})|}{\prod_{\substack{v=1\\v\neq j}}^{l}|s(\tau_{i,j}-\tau_{i,v})|}\leqslant\left(\frac{s(\bar{\Delta}_{m,k-1})}{s(\Theta_m)}\right)^{l-1}, \tag{9.4}$$

where $\{\theta_1, ..., \theta_{l-1}\}:=\{\tau_{i,1}, ..., \tau_{i,j-1}, \tau_{i,j+1}, ..., \tau_{i,l}\}$. We now estimate the size of $|R(\tau_{i,j})|$. By (4.8),

$$|R(\tau_{i,j})|\leqslant\frac{2^{\sigma-1}}{(\sigma-1)!}\int_t^{\tau_{i,j}}|s(\tau_{i,j}-y)^{\sigma-1}|\,|L_\sigma f(y)|\,dy$$
$$\leqslant\frac{2^{\sigma-1}}{(\sigma-1)!}\max_{y\in J_{t,\tau_{i,j}}}|s(\tau_{i,j}-y)^{\sigma-1}|\,\bar{\Delta}_{m,k}^{1-1/p}\,\|L_\sigma f\|_{L_p[I_m]}$$
$$\leqslant\frac{2^{\sigma-1}}{(\sigma-1)!}s(\bar{\Delta}_{m,k})^{\sigma-1}\,\bar{\Delta}_{m,k}^{1-1/p}\,\|L_\sigma f\|_{L_p[I_m]},$$

where $J_{t,\tau_{i,j}} \subseteq I_m$ is the smallest interval containing both t and $\tau_{i,j}$. Combining the bounds on $\alpha^P_{i,j}$ and $R_{i,j}$ with the inequality (9.3), we have

$$|\lambda_i R| \leqslant \frac{l2^{2r}}{(\sigma-1)!}\left(\frac{s(\bar{\Delta}_{m,k-1})}{s(\Theta_m)}\right)^{l-1} s(\bar{\Delta}_{m,k}/2)^r\, \bar{\Delta}_{m,k}^{\sigma-r-1-1/p}\, \|L_\sigma f\|_{L_p[I_m]}.$$

Inserting this in (4.15) and using (4.26), we get (9.1) in the case $q=\infty$. The result for general $1\leqslant q<\infty$ follows by integrating the qth power over the interval $[t_m, t_{m+1}]$.

We turn now to the case where $l-\sigma$ is odd. Now we use the Taylor expansion (4.12) with remainder R given by formula (4.14). Then

$$\begin{aligned} |R(\tau_{i,j})| &\leqslant \frac{2^{\sigma-1}}{(\sigma-1)!}\int_t^{\tau_{i,j}} |(s(\tau_{i,j}-y)^{\sigma-1})| \\ &\quad \times\left[\,|DL_{\sigma-1}f(y)| + \frac{\sigma}{2}\,|(s(\tau_{i,j}-y))|\,|L_{\sigma-1}f(y)|\right] dy \\ &\leqslant \frac{2^{\sigma-1}}{(\sigma-1)!}\, s(\bar{\Delta}_{m,k})^{\sigma-1}\, \bar{\Delta}_{m,k}^{1-1/p} \\ &\quad \times\left[\,\|DL_{\sigma-1}f\|_{L_p[I_m]} + \frac{\sigma}{2}\, s(\bar{\Delta}_{m,k})\, \|L_{\sigma-1}f\|_{L_p[I_m]}\right]. \end{aligned}$$

Combining this with (9.3), we get (9.2) for $q=\infty$. The result for general $1\leqslant q<\infty$ follows by integrating the qth power over the interval $[t_m, t_{m+1}]$. ∎

We conclude with a proof of Theorem 7.3. We proceed as in Section 9. In particular, if $l-\sigma$ is even, then raising (9.1) to the qth power and summing over all v such that $t_{m_v} < t_{m_v+1}$, and applying Jensen's inequality gives (7.12) with

$$K := \frac{kl(2k-1)\,2^r}{(\sigma-1)!}\left(\frac{s(\bar{\Delta}_{k-1})}{s(\Theta)}\right)^{l-1} \prod_{v=1}^{r} \frac{s(\bar{\Delta}_k)}{s(\underline{\Delta}_{k-v})}\, C_{k,r,\Delta},$$

where $C_{k,r,\Delta}$ is given in (8.16). Similarly, if $l-\sigma$ is odd, then starting with (9.2), we get (7.13).

10. MESH INDEPENDENCE

The constants appearing in both the local and global error bounds for the quasi-interpolants $Q^D_{k,l}$ and $Q^P_{k,l}$ presented in Sections 8 and 9 depend on the spacing of the knots defining the spline space. In this section we describe conditions under which this dependence can be removed for $Q^D_{k,l}$.

THEOREM 10.1. *Fix* $0 \leqslant r < \sigma \leqslant l \leqslant k$ *with* $k-l$ *even and* $2r \leqslant k$, *and suppose*

$$\tau_i \in [t_{i+r}, t_{i+k-r}], \qquad i = 1, ..., n. \tag{10.1}$$

Then the constant $K_{m,k,r,\sigma,\Delta}$ *in* (8.9) *can be replaced by*

$$K_{k,r,\sigma} := \frac{k!\, 2^{(2r+k-1)/2}}{(\sigma-1)!\,(k-r-1)!}.$$

Moreover, (6.10) *holds with the constant* $(2k-1)\, K_{k,r,\sigma}$.

Proof. We rework the proof of Theorem 8.2 using the notation introduced there. Let t be a point in the interval $[t_m, t_{m+1}]$, and fix $m+1-k \leqslant i \leqslant m$. We begin by showing that

$$\frac{\prod_{\nu=1}^{r} |s((y - t_{i+i_\nu})/2)|}{\prod_{\nu=1}^{r} |s(\Delta_{i,m,k-\nu}/2)|} \leqslant 1, \qquad y \in J_{t,\tau_i}, \tag{10.2}$$

for all $1 \leqslant i_1 < \cdots < i_r \leqslant k-1$. For each $\nu = 1, ..., r$, let $\Gamma_{k-\nu} = [t_{i+\mu}, t_{i+\mu+k-\nu}] \subset [t_i, t_{i+k}]$ be an interval of length $\Delta_{i,m,k-\nu}$ which contains $[t_m, t_{m+1}]$. Since there are only r subintervals to the left and to the right of $[t_{i+r}, t_{i+k-r}]$, it follows that $[t_{i+r}, t_{i+k-r}] \subseteq \Gamma_{k-\nu}$ for $\nu = 1, ..., r$. Then (10.1) implies that $\Gamma_{k-\nu}$ contains τ_i, and thus all points $y \in J_{t,\tau_i}$.

Since the interval $\Gamma_{k-\nu}$ contains at least $k-\nu$ of the points $\{t_{i+i_1}, ..., t_{i+i_{k-1}}\}$, it follows that the set $\Gamma_{k-\nu}$ contains at least $r-\nu+1$ of the points in $T := \{t_{i+i_1}, ..., t_{i+i_r}\}$. Thus, we can choose some $t_r^* \in T \cap \Gamma_{k-r}$, and it follows that $|y - t_r^*| \leqslant \Delta_{i,m,k-r}$. Proceeding inductively, we can now choose points $t_{r-1}^*, ..., t_1^*$ with $t_\nu \in T \cap \Gamma_{k-\nu}$ such that

$$|y - t_\nu^*| \leqslant \Delta_{i,m,k-\nu}, \qquad \nu = 1, ..., r.$$

Now the fact that $s(x)$ is monotone increasing for $0 \leqslant x \leqslant \pi$ implies (10.2).

The proof for the local error bound in the uniform norm $q = \infty$ now follows from (10.2) and (4.28). The result for general $1 \leqslant q \leqslant \infty$ then follows immediately. The global result is established with Jensen's inequality in exactly the same way as in Theorem 8.2, leading to the extra factor $2k-1$. ∎

We note that the hypothesis $2r \leqslant k$ is needed to ensure that the interval (10.1) is nonempty, and so the above mesh-independent error bound works only for derivatives up to order $r \leqslant k/2$.

11. REMARKS

Remark 11.1. The quasi-interpolants $Q^D_{k,l}$ and $Q^P_{k,l}$ discussed in this paper can be considered to be extreme cases of a more general class of quasi-interpolants which are based on the linear functionals

$$\lambda_{i,j} := D_{k,\nu_j} f(\tau_{i,j}), \qquad j = 1, ..., l,$$

where $\tau_{i,1} \leqslant \cdots \leqslant \tau_{i,l}$ are prescribed, and

$$\nu_j := \max\{\mu : \tau_{i,j-\mu} = \cdots = \tau_{i,j}\},$$

for all $i = 1, ..., n$. $Q^D_{k,l}$ corresponds to taking all the $\tau_{i,j} = \tau_i$, while $Q^P_{k,l}$ corresponds to selecting $\tau_{i,1} < \cdots < \tau_{i,l}$. The analysis of these more general quasi-interpolants can be based on the trigonometric Newton form in [7], and will be presented in a separate paper.

Remark 11.2. The spline space $\mathscr{S}$ defined in Section 1 was defined on an extended knot sequence (2.1) which stacks a total of k knots at each of the endpoints of an interval J. However, the entire analysis works equally well if we extend the knots so that $t_1 \leqslant \cdots \leqslant t_k \leqslant a$ and $b \leqslant t_{n+1} \leqslant \cdots \leqslant t_{n+k}$. Moreover, similar results can also be established for spaces of *periodic splines* (see [10]) which are based on knots which are periodic.

Remark 11.3. The quasi-interpolants studied in this paper can be immediately applied to create multivariate quasi-interpolants by taking tensor products (cf. [8] for the polynomial spline case). In fact, we can use trigonometric quasi-interpolants in some variables, and polynomial quasi-interpolants in others (see [11] for a useful example based on the quasi-interpolant presented in (7.10)).

Remark 11.4. We have presented error bounds for functions f in the usual Sobolev spaces $L^\sigma_p[J]$. They depend on the p-norm of certain σ-order derivatives of f. As in the polynomial case [8], it is also possible to present error bounds in terms of moduli of smoothness of appropriate differential operators applied to f. They can be obtained from the trigonometric Taylor expansions.

Remark 11.5. The error bounds given here involve powers of $\bar{\Delta}$ which are the same as those obtained for best approximation by trigonometric splines (cf. [3–5, 7]). In other words, these *linear* quasi-interpolation operators give best orders of approximation.

Remark 11.6. There is a certain arbitrariness in the way in which we defined the basic functions $s(x)$ and $c(x)$ at the beginning of Section 2. In fact, everything we have done here would work equally well if we set

$c(x) = \sin(\alpha x)$ and $s(x) = \cos(\alpha x)$, where α is an arbitrary positive real number, cf. [6], although of course the constants in the various error bounds change. With this choice of s and c, it is interesting to note that as $\alpha \to 0$, the trigonometric B-splines converge to the usual polynomial B-splines, and the quasi-interpolants constructed here converge to their polynomial analogs as discussed in [1, 8].

Remark 11.7. Our proof of error bounds for $Q^P_{k,l}$ was based on bounding the coefficients $\alpha^P_{i,j}$ which appear in (7.4). Instead of using dual polynomials, as was done in Section 9, we could also have followed the approach used in Section 8 for $Q^D_{k,l}$ by identifying the $\alpha_{i,j}$ as blossoms of certain coefficients appearing in the trigonometric Taylor expansion.

Remark 11.8. In Section 9 we have shown that under certain conditions on the mesh, the constants in our error bound for $Q^D_{k,l}$ do not depend on mesh ratios, at least for derivatives of order $r \leqslant k/2$. By working with the divided difference definition of trigonometric B-splines, it is possible to improve these results somewhat as was done in [8] for the polynomial case. Moreover, the divided difference approach also leads to mesh-independence results for $Q^P_{k,l}$, and even for the more general quasi-interpolants described in Remark 11.1. We do this in a separate paper.

Remark 11.9. Given an arbitrary mesh, it is possible to establish error bounds where the constants are independent of the mesh for *all* derivatives if we first *thin out* the mesh using the technique described in Lemma 6.17 of [10].

Remark 11.10. It is also possible to define trigonometric spline quasi-interpolants based on local integral functionals. We discuss them in a separate paper.

Remark 11.11. There are interesting dual forms for the Taylor expansions given in Lemmas 4.1 and 4.3. In particular, if we write the factors of L_σ in (4.8) in reverse order and then perform the integration by parts, we get the alternate form

$$U_{\sigma,t} f(x) = \frac{2^{\sigma-1}}{(\sigma-1)!} \sum_{j=1}^{\sigma} M_{\sigma,\sigma-j}[s(x-t)^{\sigma-1}]\, D_{\sigma,j-1} f(t)$$

for the Taylor expansion of Lemma 4.1. Similarly, we have the alternative form

$$\tilde{U}_{\sigma,t} f(x) = \frac{2^{\sigma}}{\sigma!} \sum_{j=1}^{\sigma} M_{\sigma+1,\sigma-j+1}[s(x-t)^{\sigma}]\, D_{\sigma+1,j-1} f(t)$$

for the Taylor expansion of Lemma 4.3.

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